April 13, 1981

Dr. Robert L. Schiffman
Department of Civil Engineering
Campus Box 428
University of Colorado
Boulder, Colorado 80309

Dear Bob:

   Thank you very much for sending me a copy of your report assessing my research project for the U.S.G.S. There is no way I can rebuttal you point by point in the time you allotted me since I have been out of the office most of the month of March. However, I have no objection to you sending in your report now and I'll send my rebuttal to you and the U.S.G.S. as soon as I finish. By the way, since our meeting in Reston, Virginia I have gone through the literature again looking for some hard experimental data that proves the effective stress assumption and I still can find none. If you can locate such information or if you know of some unpublished hard data I would appreciate locating it.

                              Sincerely,

                              Louis J. Thompson
                              Professor

cc: John Gregory, U.S.G.S.

LJT/jnt

THE EFFECTIVE STRESS CONCEPT

by

Robert L. Schiffman

Geotechnical engineering is based upon the concept of effective stresses. This concept is partly geometric and partly physical. It serves to isolate the deformation producing mechanisms which occur when a geologic material is subjected to a disturbance (Schiffman, 1970).

This report reviews current theories of effective stress. Relationships between competing and complementary theories are analyzed.

THE MIXTURE

The mixture (interacting continua) is a porous system consisting of (m) interacting continua $(c^{(s)})$, where (s) takes on values of $(1, 2, \ldots, m)$. Each continuum $(c^{(s)})$ is in relative motion with respect to a fixed coordinate system (Atkin and Craine, 1976a, 1976b). The elementary bulk volume of the porous mass (δV) is composed of representative component volumes $(\delta V^{(s)})$. Thus

$$\delta V = \sum_{s=1}^{m} \delta V^{(s)} . \tag{1}$$

The volume fraction $(n^{(s)})$ of each continuum is defined as

$$n^{(s)} = \frac{\delta V^{(s)}}{\delta V} . \tag{2}$$

If the mixture is defined as containing two phases, a solid (1), and voids (2), the porosity of the mixture is defined as

$$n = \frac{\delta V^{(2)}}{\delta V} . \tag{3}$$

The state of stress of the mixture can be defined in several ways depending on the reference volumes used.

Force Components

A set of boundary forces and/or displacements applied to the bulk mixture will produce an internal relative force vector point function (δF_i). This total force is the sum of the component forces for each continuum $(\delta F_i^{(s)})$ such that

$$\delta F_i = \sum_{s=1}^{m} \delta F_i^{(s)} . \tag{4}$$

The tractions and stress components are defined by the relationship between the elementary forces and elementary volumes.

Stress Components

The total traction of the mixture (T_i) is defined in terms of the total force per unit of total surface of the mixture. Thus

$$T_i = \lim_{\delta S \to 0} \frac{\delta F_i}{\delta S} , \tag{5}$$

where (δS) is the surface cut across the total elementary volume (δV). It is noted that in this treatment the elementary surface (δS) is used interchangeably with the elementary volume (δV).

The total stress tensor (τ_{ij}) represents the components of the total traction. Thus

$$T_i = n_j \tau_{ij} \quad , \tag{6}$$

where (n_j) is the unit vector normal to the surface (δS).

A component traction $(T_i^{(s)})$ can be defined as

$$T_i^{(s)} = \lim_{\delta S \to 0} \frac{\delta F_i^{(s)}}{\delta S} \quad , \tag{7}$$

and a component stress tensor $(\bar{\sigma}_{ij}^{(s)})$ is then

$$T_i^{(s)} = n_j \bar{\sigma}_{ij}^{(s)} \quad . \tag{8}$$

The stress tensor $(\bar{\sigma}_{ij}^{(s)})$ is called the "partial stress" in the mixture theory literature (Atkin and Craine, 1976a). The soil mechanics literature calls this entity the "intergranular stress" since it is generally used to describe the stresses acting at the mineral contacts in a soil mass (Skempton, 1961).

The total and partial stresses are related by virtue of equation (4) by

$$\tau_{ij} = \sum_{s=1}^{m} \bar{\sigma}_{ij}^{(s)} \quad . \tag{9}$$

The component tractions can be defined in terms of their own surface areas. Thus a traction $(t_i^{(s)})$ can be defined as

$$t_i^{(s)} = \lim_{\delta S^{(s)} \to 0} \frac{\delta F_i^{(s)}}{\delta S^{(s)}} \quad . \tag{10}$$

The companion stress tensor is

$$t_i^{(s)} = n_j \sigma_{ij}^{(s)} \quad .$$

(11)

This stress is referred to as a "pore stress" since it is used to describe the state of stress of the interpore material in a soil or rock mass.

The various stress components can be related by

$$\bar{\sigma}_{ij}^{(s)} = n^{(s)} \sigma_{ij}^{(s)} \quad ,$$

(12)

and then

$$\tau_{ij} = \sum_{s=1}^{m} n^{(s)} \sigma_{ij}^{(s)} \quad .$$

(13)

These are purely geometric relationships, which must be modified and interpreted in order to describe the behavior of a complete porous soil or rock mass.

Strain Components

The total mass density of the mixture is (ρ). The partial densities ($\rho^{(s)}$) are the masses of the (s) constituents per unit of the mass volume. Thus

$$\rho = \sum_{s=1}^{m} \rho^{(s)} \quad .$$

(14)

The partial strain tensors, ($e_{ij}^{(s)}$) if infinitesimal strain theory is assumed, are (Steel, 1967)

$$e_{ij}^{(s)} = \frac{1}{2}\left[\frac{\partial u_i^{(s)}}{\partial x_j^{(s)}} + \frac{\partial u_j^{(s)}}{\partial x_i^{(s)}}\right] , \qquad (15)$$

where $(u_i^{(s)})$ are the partial displacement vectors and $(x_i^{(s)})$ are the partial length vectors. It is noted that the theory of mixtures assumes that all the interacting continua occupy the same space at the same time. Thus, for purposes of strain definition, there need not be a positional distinction between phases.

The partial dilatation $(e^{(s)})$ is defined as[1]

$$\rho^{(s)} = \rho_0^{(s)}(1 - e^{(s)}) , \qquad (16)$$

where $(\rho_0^{(s)})$ is the partial equilibrium mass density.

Stress-Strain Relationships

The stress-strain relationships can be obtained directly by the Helmholtz free energy relationships. If the mixture contains two isotropic elastic solids, deforming under isothermal conditions, the stress-strain relationships are (Steel, 1968).

$$\bar{\sigma}_{ij}^{(1)} = \lambda^{(1)}e^{(1)}\delta_{ij} + 2\mu^{(1)}e_{ij}^{(1)} + \bar{\lambda}e^{(2)}\delta_{ij} + 2\bar{\mu}e_{ij}^{(2)} , \qquad (17a)$$

$$\bar{\sigma}_{ij}^{(2)} = \lambda^{(2)}e^{(2)}\delta_{ij} + 2\mu^{(2)}e_{ij}^{(2)} + \bar{\lambda}e^{(1)}\delta_{ij} + 2\bar{\mu}e_{ij}^{(1)} , \qquad (17b)$$

where $(\lambda^{(1)})$, $(\mu^{(1)})$, $(\lambda^{(2)})$ and $(\mu^{(2)})$ are Lamé's constants for materials (1) and (2), respectively, when the materials are separated. The coefficients $(\bar{\lambda})$ and $(\bar{\mu})$ represent the interaction between the materials.

---

[1] It is noted that a different definition of the partial dilation $(e^{(s)})$ is given elsewhere (Garg and Nur, 1973). This definition bases the dilation upon the current partial mass density $(\rho^{(s)})$ instead of the initial mass density $(\rho_0^{(s)})$ as given by equation (16).

It is noted that these stress-strain relationships assume that there are no partial initial stresses.

EFFECTIVE AND INTERGRANULAR STRESSES

In the discussion which follows it will be assumed that the porous mass consists of two phases, a solid (or granular) phase and a pore phase. Furthermore, the pores will be saturated with a liquid. The sign convention will be that of the theory of elasticity in which tension will carry a positive sign. The liquid will be assumed to sustain no shear stresses. The intergranular stress relating to the solid phase (1) will be ($\bar{\sigma}_{ij}$). The pore stress relating to the pore phase (2) will be (σ). In conformity with the sign convention the pore pressure (p) is

$$\sigma = -p \, , \qquad\qquad (18a)$$

and

$$\bar{\sigma} = -np \, . \qquad\qquad (18b)$$

Using the above sign convention and notation the relationships between total, intergranular, and pore stresses will be developed. The concept of effective stress will also be developed.

Intergranular Stress

Equation (9), for the fluid saturated two-phase medium is

$$\tau_{ij} = \bar{\sigma}_{ij} + \sigma\delta_{ij} \, , \qquad\qquad (9bis)$$

in terms of the partial stress, or

$$\tau_{ij} = \bar{\sigma}_{ij} - np\delta_{ij} \quad , \tag{19}$$

in terms of the intergranular stress for the mineral skeleton and the pore pressure of the interpore liquid.

Effective Stress

The concept of an effective stress is defined (Terzaghi, 1942) as follows:

> "...the compressive stress in a saturated soil consists of two parts with very different mechanical effects. One part which is equal to the pressure in the water produces neither a measurable compression nor a measurable increase of the shearing resistance. This part is called the neutral stress (p)...."

> "The second part (σ'_{ij}) of the total stress (τ_{ij}) is equal to the difference between the total stress and the neutral stress (p). This second part

$$\sigma'_{ij} = \tau_{ij} + p\delta_{ij} \tag{20}$$

> is called the effective stress, because it represents that part of the total stress which produces measurable effects such as compaction or an increase of the shearing resistance."[2]

It is noted that equations (19) and (20) differ by a factor (n). This implies that the intergranular stress ($\bar{\sigma}_{ij}$) is not the effective stress (σ'_{ij}). Furthermore, the above definition assumes an incompressible pore fluid. All of these factors can be taken into consideration by a formulation of the effective stress equation (Schiffman 1970)

---

[2]This definition has been editorialized to conform to the notation and sign convention of this paper.

$$\sigma'_{ij} = \tau_{ij} + \alpha p \delta_{ij} \quad , \tag{21a}$$

where (α) is defined as a "porous medium interaction coefficient", with a range of values

$$n \leq \alpha < 1 \quad . \tag{21b}$$

There are a variety of mutually compatible interpretations of the coefficient (α).

COMPRESSION OF A POROUS MASS

Assume that the porous mass consists of an isotropic, elastic solid and a compressible, Newtonian fluid. Further, assume that the mass is subjected to a set of boundary forces which instigate a deformation. Energy arguments lead to the following relationship between the intergranular stresses and the strain of the solid and fluid portions of the porous mass (Biot, 1956; Biot and Willis, 1957)

$$\bar{\sigma}_{ij} = Ae\delta_{ij} + 2Ne_{ij} + Q\epsilon\delta_{ij} \quad , \tag{22a}$$

$$\bar{\sigma} = Qe + R\epsilon \quad , \tag{22b}$$

where (A), (N), (Q), (R) are elastic constants of the porous mass, (e_{ij}) is the strain of the solid portion of the porous medium, (e) is the dilatation of the porous medium and (ϵ) is the dilatation of the pore fluid.

Equations (22) are fully compatible with equations (17) recognizing that (A) and (N) are Lamé's constants of the solid, (R) is the bulk modulus of the fluid, and (Q) is the interaction bulk modulus.

It is usually assumed that the soil and rock grains in a porous medium are by themselves incompressible. The deformation of the "solid"

is thus the deformation of the porous skeleton which encompasses the motions of the mineral grains and the rearrangement of grains from one state of packing to another.

This interpretation is fully consistent with the theory of mixtures in which the porous mass is treated as a continuum.

The effective stress principle as stated by Terzaghi (1942) implies that, at least for a soil saturated with an incompressible fluid, the effective stresses are the deformation producing entities. Thus the isotropic, elastic stress-strain relationship is

$$\sigma'_{ij} = \lambda' e \delta_{ij} + 2\mu' e_{ij} \quad , \tag{23}$$

where (λ') and (μ') are the effective Lamé's constants.

Relationship Between Effective and Intergranular Stresses

Repeating equations (19) and (21a) provides

$$\tau_{ij} = \bar{\sigma}_{ij} - np\delta_{ij} \quad , \tag{19}$$

$$\tau_{ij} = \sigma'_{ij} - \alpha p\delta_{ij} \quad . \tag{21a bis}$$

Solving for the effective stress by eliminating the total stress results in

$$\sigma'_{ij} = \bar{\sigma}_{ij} - (n - \alpha)p\delta_{ij} \quad . \tag{24}$$

It is noted that when ($\alpha = n$), the effective stress is the intergranular stress.

Relationship Between Elastic Constants

Substituting the stress-strain relationships (22) into equation (24) results in

$$\sigma'_{ij} = \left[A + Q \left(1 - \frac{\alpha}{n} \right) \right] e\delta_{ij} + 2Ne_{ij} + \left[Q + R \left(1 - \frac{\alpha}{n} \right) \right] \varepsilon\delta_{ij} \quad . \qquad (25)$$

Comparing equations (23) and (25)

$$\lambda' = A - \frac{Q^2}{R} \quad , \qquad (26a)$$

$$\alpha = n \left[\frac{Q + R}{R} \right] , \qquad (26b)$$

$$\mu' = N \quad . \qquad (26c)$$

The stress-strain relationsp then become

$$\tau_{ij} = \lambda_u e\delta_{ij} + 2\mu' e_{ij} - \frac{\alpha R}{n} (e - \varepsilon)\delta_{ij} \quad , \qquad (27a)$$

$$p = \frac{R}{n^2} \left[n(e - \varepsilon) - \alpha e \right] \quad , \qquad (27b)$$

where

$$\lambda_u = \lambda' + \frac{\alpha^2 R}{n^2} \quad , \qquad (27c)$$

is the undrained Lamé constant.

Performing jacketed and unjacketed tests on the porous mass results in

$$\alpha = 1 - \frac{\bar{\beta}}{\beta} \quad , \qquad (28)$$

where $(\bar{\beta})$ is the unjacketed compressibility of the solid material and (β) is the jacketed compressibility of the porous mass. This relationship is fully in accordance with the relationship developed by Skempton (1961).

CONCEPTS OF EFFECTIVE STRESS

The discussion above established the effective stress equation as

$$\sigma'_{ij} = \tau_{ij} + \alpha p \delta_{ij} \quad . \tag{21a}$$

with various interpretations of the coefficient (α).

Compressibility Interpretation

Several authors (Biot and Willis, 1957; Geertsma, 1957; Skempton, 1961; Nur and Byerlee, 1971) have shown that when the porous mass is assumed to be an isotropic, elastic medium, the coefficient (α) is governed by equation (28). This relationship is fully consistent with mixture theory and thus is solidly based in continuum mechanics. A minor modification has suggested the form (Suklje, 1969).

$$\alpha = 1 - \frac{n\bar{\beta}}{\beta} \quad . \tag{29}$$

This interpretation of (α) provides a range from zero, for a non-porous mass, to unity for a very porous mass or a mass in which the fluid is incompressible. Its value will generally be close to unity for most materials due to the relative compressibilities of particles and mass.

Schiffman (1970) suggested that (α) ranges from the porosity (n) to unity. This geometric interpretation is fully in accord with other interpretations since the compressibility ratio $(\bar{\beta}/\beta)$ is directly related to the porosity. A nonporous mass will have a porosity of zero.

Some question has been raised as to the applicability of the effective stress equation (21a) in a nonlinear regime (Garg and Nur, 1973; Morita and Gray, 1980a, 1980b). There is reason to believe that the relationships developed above will hold for incremental linear behavior within a totality of nonlinear response.

Shear Strength Interpretations

Skempton (1961) has proposed that for shear strength

$$\alpha = 1 - \frac{a \tan \psi}{\tan \phi'} \quad , \tag{30}$$

where (a) is the area of contact between particles per unit gross area of the material, (ψ) is the angle of intrinsic friction of the solid and (ϕ') is the effective angle of internal friction of the mass. It was concluded that (α) is effectively unity for soil, but may not be so when the material is saturated rock or concrete. In these cases, however, the assumption that ($\alpha = 1$) is a reasonable first approximation.

A General Concept

A general principle of effective stress which has been postulated for partly saturated soils (Blight, 1965) is:

> "...the effective stress is that function of total stress and pore pressure which controls the mechanical effects of a change in stress.... The principle of effective stress is the assertion that such a function exists with parameters which are determinate under a given set of conditions."

This can be formulated as

$$\sigma'_{ij} = \tau_{ij} + f(p) \quad , \tag{31}$$

where the function (f) relates to (α) and any of the nonlinearities involved.

REFERENCES

Atkin, R.J. and Craine, R.E., (1976a), "Continuum Theories of Mixtures: Basic Theory and Historical Development," Quarterly Journal of Mechanics and Applied Mathematics, <u>29</u>, pp. 209-244.

Atkin, R.J. and Craine, R.E., (1976b), "Continuum Theories of Mixtures: Applications," Journal of the Institute of Mathematics and Its Applications, <u>17</u>, pp. 153-207.

Biot, M.A., (1956), "General Solutions of the Equations of Elasticity and Consolidation for a Porous Material," Journal of Applied Mechanics, <u>23</u>, pp. 91-96.

Biot, M.A. and Willis, D.G., (1957), "The Elastic Coefficients of Consolidation," Journal of Applied Mechanics, <u>24</u>, pp. 594-601.

Blight, G.E., (1965), "A Study of Effective Stresses for Volume Change," <u>Moisture Equilibria and Moisture Changes in Soils,</u> (Edited by G.D. Aitchison), Butterworths, Australia, pp. 259-269.

Garg, S.K. and Nur, A., (1973), "Effective Stress Laws for Fluid-Saturated Porous Rocks," Journal of Geophysical Research, <u>78</u>, pp. 5911-5921.

Geertsma, J., (1957), "The Effect of Fluid Pressure Decline on Volumetric Changes of Porous Rocks," Transactions of AIME, <u>210</u>, pp. 331-340.

Morita, N. and Gray, K.E., (1980a), "A Constitutive Equation for Nonlinear Stress-Strain Curves in Rocks and Its Application to Stress Analysis Around a Borehole During Drilling," Society of Petroleum Engineers of AIME, Paper SPE 9328.

Morita, N. and Gray, K.E., (1980b), "Effective Stress Concepts, Definitions and Restrictions," *Private commnication.*

Nur, A. and Byerlee, J.D., (1972), "An Exact Effective Stress Law for Elastic Deformation of Rock with Fluids," Journal of Geophysical Research, <u>76</u>, pp. 6414-6419.

Schiffman, R.L., (1970), "The Stress Components of a Porous Medium," Journal of Geophysical Research, <u>75</u>, pp. 4035-4038.

Skempton, A.W., (1961), "Effective Stress in Soils, Concrete and Rocks," <u>Pore Pressure and Suction in Soils,</u> Butterworths, London, United Kingdom, pp. 4-16.

Steel, T.R., (1967), "Applications of a Theory of Interacting Continua," Quarterly Journal of Mechanics and Applied Mathematics, 20, pp. 57-72.

Steel, T.R., (1968), "Determination of the Constitutive Coefficients for a Mixture of Two Solids," International Journal of Solids and Structures, 4, pp. 1149-1160.

Suklje, L., (1969), Rheological Aspects of Soil Mechanics, Wiley-Interscience, London, United Kingdom.

Terzaghi, K., (1942), Theoretical Soil Mechanics, John Wiley and Sons, Inc., New York, New York.